

ON NEATLY ATOMIC CYLINDRIC SET ALGEBRAS

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Abstract

For a cylindric set algebra \mathfrak{A} with (base U and) unit 0U and $X \subseteq U$, let \mathfrak{A}_X be the subalgebra of $\wp({}^0U)$ generated by $A \cup \{\{u\} \times {}^0U : u \in X\}$. $At\mathfrak{D}$ denotes the set of atoms of \mathfrak{D} . It is shown that, there exists a simple and countable $\mathfrak{A} \in \mathbf{Lf}_\omega$ such that \mathfrak{A}_n is atomic for every n and the following hold. For every $\mathfrak{C} \in \mathbf{Lf}_\omega \cap \mathbf{Cs}_\omega^{reg}$, $\mathfrak{C} \cong \mathfrak{A}$ and \mathfrak{C} has base U , there is a non-empty $V \subseteq U$ such that for any $X \subseteq V$, $\mathfrak{A}_n(\mathfrak{C}_X)$ is atomic for every n . Furthermore, $\bigcup \{x : x \in At\mathfrak{A}_n(\mathfrak{C}_X)\} = 1$ for every n , if and only if $|X| \leq \aleph_1$.

1. Introduction

This paper belongs to (Tarskian) algebraic logic [10], [11]. Similar investigations were carried out by Biro and Shelah [3], but here we give a different construction. We follow standard notation, that adopted in [5]. Let α be an ordinal. \mathbf{CA}_α stands for the class of cylindric algebras of

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dimension α . \mathbf{Lf}_α is the class of locally finite \mathbf{CA}_α 's. \mathfrak{A} is such, if $\Delta x = \{i \in \alpha : c_i x \neq x\}$ is finite for every $x \in A$. If \mathcal{L} is a first order language (with α many variables) and T an \mathcal{L} theory, then $\mathfrak{Fm}^{\mathcal{L}}/T$ denotes the Tarski-Lindenbaum quotient cylindric algebra of formulas. $\mathfrak{Fm}^{\mathcal{L}}/T \in \mathbf{Lf}_\alpha$. This reflects the simple fact that first order formulas contain finitely many variables. In fact, locally finite algebras, and algebras of formulas are in one to one correspondance, in some (exact) sense [7] 4.3.28. For $\mathfrak{A} \in \mathbf{CA}_\alpha$ and $n < \alpha$, then $\mathfrak{Nr}_n \mathfrak{A}$ [5] 2.6.28 denotes the n -neat reduct of \mathfrak{A} . $\mathfrak{Nr}_n \mathfrak{A}$ has universe $\{x \in A : \Delta x \subseteq n\}$ and operations induced from \mathfrak{A} up to the index n . It is absolutely straight forward to show that $\mathfrak{Nr}_n \mathfrak{A} \in \mathbf{CA}_n$. For a formula algebra \mathfrak{F} , $\mathfrak{Nr}_n \mathfrak{F}$ is the n -dimensional cylindric algebra, where only formulas with n free variables are allowed. \mathbf{Cs}_α and \mathbf{Cs}_α^{reg} stand for the classes of cylindric set algebras and regular cylindric set algebras of dimension α , respectively. The greatest element of such algebras is a set of the form ${}^\alpha U$. The cylindric (non-boolean) operations are defined for $X \subseteq A$ and $i, j < \alpha$ by

$$C_i X = \{s \in {}^\alpha U : \exists t \in X : t(j) = s(j), \forall j \neq i\},$$

and

$$D_{ij} = \{s \in {}^\alpha U : s_i = s_j\}.$$

Let \mathfrak{M} be a model of a first order language \mathcal{L} and $\phi^{\mathfrak{M}}$ be the set of ω -ary assignments satisfying ϕ in \mathfrak{M} . Then,

$$\mathbf{Cs}_\omega^{\mathfrak{M}} = \{\phi^{\mathfrak{M}} : \phi \in \mathcal{L}\}$$

is the typical (universe of) cylindric set algebra obtained by defining the extra non-boolean operations by

$$C_i \phi^{\mathfrak{M}} = \exists x_i \phi^{\mathfrak{M}} \quad \text{and} \quad D_{ij} = (x_i = x_j)^{\mathfrak{M}},$$

$\mathfrak{A} \in \mathbf{Cs}_\alpha^{reg}$, if $\mathfrak{A} \in \mathbf{Cs}_\alpha$ with greatest element ${}^\alpha U$, say, such that for $f, g \in {}^\alpha U$, if $f \upharpoonright \Delta x = g \upharpoonright \Delta x$, then $f \in x$ iff $g \in x$. Regularity reflects the fact that, if two assignments coincide on the free variables in a formula, then both satisfy the formula or none does. The class $\mathbf{Cs}_\omega^{reg} \cap \mathbf{Lf}_\omega$ corresponds to the class of models. Any \mathbf{Cs}_ω^{m} is in this class, and conversely every algebra in $\mathbf{Cs}_\omega^{reg} \cap \mathbf{Lf}_\omega$ defines a model in a natural way. Several theorems in first order logic are reflected algebraically by the class \mathbf{Lf}_ω , or rather the interaction of this class with the class \mathbf{Cs}_ω^{reg} . Completeness, for example, is reflected by the fact that every locally finite algebra of dimension ω is representable, i.e., isomorphic to a subdirect product of regular set algebras (of the same dimension). The omitting types theorem is reflected by the fact that such representations can be chosen to preserve given infinitary meets.¹ A consequence of the omitting types theorem is the existence of atomic or prime models for countable atomic theories. One algebraic counterpart [12] is

Theorem 1. *Let \mathfrak{A} be a locally finite, countably generated, cylindric algebra of dimension ω such that for every $n \in \omega$, $\mathfrak{Nr}_n \mathfrak{A}$ is atomic. If \mathfrak{B}_i are regular set algebras f_i is an isomorphism of \mathfrak{A} onto $\mathfrak{B}_i (i \in 2)$, then $f_1 \circ f_0^{-1}$ is a lower-base isomorphism.*

An algebra \mathfrak{A} , such that $\mathfrak{Nr}_n \mathfrak{A}$ is atomic for every $n \in \omega$ is called *neatly atomic*. A lower base isomorphism is the composition of a strong ext-isomorphism with a sub-base isomorphism in the terminology of [7] 3.1.41. Another algebraic expression of existence of prime models for the countable case is:

¹ The number of types omitted can be $< \text{cov } K$, where $\text{cov } K$ is the least cardinal λ such that the real line can be covered by λ nowhere dense sets. $\omega < \text{cov } K \leq {}^\omega 2$.

Theorem 2. *Let $\mathfrak{A} \in \mathbf{Lf}_\omega$ be countable and neatly atomic. Then, there exists $\mathfrak{B} \in \mathbf{Cs}_\omega^{reg} \cap \mathbf{Lf}_\omega$ such that $\mathfrak{A} \cong \mathfrak{B}$, and $\bigcup \{x : x \in \text{At}\mathfrak{N}_n \mathfrak{B}\} = 1$. Furthermore, $\mathfrak{C} \cong \mathfrak{B}$ implies that \mathfrak{C} is base-isomorphic to some \mathfrak{D} , which is sub-isomorphic to \mathfrak{B} .*

Results resembling Theorems 1 and 2 occur in [6], where the notion of lower base isomorphism is introduced. Now, let $\mathfrak{A} \in \mathbf{Cs}_\omega^{reg} \cap \mathbf{Lf}_\omega$ with base U and let $V \subseteq U$. Then \mathfrak{A}_V is the subalgebra of $\wp({}^\omega U)$ generated by $A \cup \{\{u\} \times {}^\omega U : u \in V\}$. It is clear that $\mathfrak{A}_V \in \mathbf{Cs}_\omega^{reg} \cap \mathbf{Lf}_\omega$.

We shall prove the following:

Theorem 3. *There exists $\mathfrak{A} \in \mathbf{Lf}_\omega$ such that \mathfrak{A} is countable, simple, and neatly atomic. For every $\mathfrak{C} \in \mathbf{Lf}_\omega \cap \mathbf{Cs}_\omega^{reg}$ with base U , if $\mathfrak{C} \cong \mathfrak{A}$, there is a non-empty $V \subseteq U$ such that for any $X \subseteq V$, \mathfrak{C}_X is neatly atomic. Furthermore, $\bigcup \{x : x \in \text{At}\mathfrak{N}_n \mathfrak{C}_X\} = 1$ for every n , if and only if $|X| \leq \aleph_1$.*

2. Proof

We assume as known some (basic) model theoretic notions, like atomic theories, atomic models, and indiscernibles. A good source is [8]. Let \mathcal{L} be a relational first order language with unary predicates U and V , a relation symbol p and countable collections of ternary relation symbols f_n and binary relation symbols R_n for every $n \in \omega$. For a subset X of an L structure M , define the closure of X in M $\text{cl}(X)$ to be the transitive closure of $\text{cl}_0(X) = \{t : f_n(b, c, t), b, c \in X, n \in \omega\}$. Let K be the class of all finite L structures \mathfrak{A} satisfying:

- (i) U and V are disjoint sets, whose union is the universe A ;
- (ii) $p \subseteq U \times V$;

- (iii) each $f_n \subseteq U \times U \times U$;
- (iv) for all $u \in W$, there exists unique $v \in V$ such that $p(u, v)$;
- (v) for all $u, w \in W$, there exists unique $v \in U$ such that $f_n(u, w, v)$;
- (vi) for each n , $R_n(x, y) \Rightarrow U(x) \wedge U(y)$;
- (vii) the family $\{R_n : n \in \omega\}$ partitions 2U into disjoint pieces;
- (viii) for each n and $m \geq n$, $R_n(x, y) \Rightarrow f_m(x, y, x)$;
- (ix) if $x', y' \in \text{cl}\{x, y\}$ and $R_n(x, y)$, then $\bigvee_{j \leq n} R_j(x', y')$;
- (x) there is no cl-independent subset of U of size 3. That is, for all $x_0, x_1, x_2 \in U$, there is a permutation σ of $\{0, 1, 2\}$ such that $x_{\sigma(0)} \in \text{cl}\{x_{\sigma(1)}, x_{\sigma(2)}\}$.

It is easy to check that K is closed under substructures and isomorphism and that K contains only countably many isomorphism types. Moreover, K has the joint embedding property and the amalgamation property. We only verify amalgamation since the proofs are similar. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K$ with $A \subseteq B$ and $A \subseteq C$ and $A = B \cap C$. It suffices to find $\mathfrak{D} \in K$ with universe $B \cup C$ such that $\mathfrak{B} \subseteq \mathfrak{D}$ and $\mathfrak{C} \subseteq \mathfrak{D}$. Let $\{b_0, \dots, b_{l-1}\}$ enumerate $U^{\mathfrak{B}}$ and $\{c_0, \dots, c_{m-1}\}$ enumerate $U^{\mathfrak{C}}$. Let $k > l, m$ be large enough such that

$$\bigcup \{R_j^{\mathfrak{B}} : j < k\} = (U^{\mathfrak{B}})^2 \quad \text{and} \quad \bigcup \{R_j^{\mathfrak{C}} : j < k\} = (U^{\mathfrak{C}})^2.$$

For each $b \in B \setminus A$ and $c \in C \setminus A$, let $f_j(b, c, c_j)$ for $j < m$ and $f_j(b, c, b)$ for $j \geq m$ and let $R_k(b, c)$ hold. Similarly, for $j < l$, let $f_j(c, b, b_j)$ and $f_j(c, b, c)$ for all $j \geq l$ and $R_k(c, b)$. Then $\mathfrak{D} \in K$. It follows there is a countable K -generic L structure \mathfrak{M} or a Fraisse limit [8]. That is,

(*) \mathfrak{M} is the union of an increasing chain of elements of K ,

(**) every element of K isomorphically embeds in \mathfrak{M} ,

(***) if $j : \mathfrak{A} \rightarrow \mathfrak{A}'$ is an isomorphism between finite substructures of \mathfrak{M} , then there is an automorphism of \mathfrak{M} extending j .

We have $V^{\mathfrak{M}}$ is infinite as there are elements of \mathfrak{A} of K with arbitrary large $V^{\mathfrak{A}}$. $V^{\mathfrak{M}}$ is set of indiscernibles because of property (***) , and the fact that any two n tuples of distinct elements of $V^{\mathfrak{M}}$ are universes of isomorphic substructures of \mathfrak{M} . Every finite subset of \mathfrak{M} is contained in an element of K , it follows that $\text{cl}(X)$ is finite for all finite subsets X of \mathfrak{M} , and there is no cl-independent subset of $U^{\mathfrak{M}}$ of size 3. Let T be the theory of \mathfrak{M} , then T is an atomic theory and \mathfrak{M} is an atomic model for this theory by property (***) . Let $\mathfrak{A} = \mathfrak{F}_m^L / T$, then $Nr_n \mathfrak{A}$ is atomic for all n and \mathfrak{A} is isomorphic to $\mathbf{Cs}_\omega^{\mathfrak{M}}$. We have our \mathfrak{A} . To achieve more, we need

Lemma 1. *Let \mathfrak{N} be a model of T , and let $X \subseteq V^{\mathfrak{N}}$. Then, the principal types are dense over X .*

Proof. Let $\theta(\bar{x}, \bar{a})$ be any consistent formula, where \bar{a} is a tuple of k distinct elements of X . Let \bar{b} be any k tuple of distinct elements from $V^{\mathfrak{M}}$. As the elements from V are indiscernible, $\mathfrak{M} \models \exists \bar{x} \theta(\bar{x}, \bar{b})$. Let \bar{c} from \mathfrak{M} realize $\theta(\bar{x}, \bar{b})$. Since \mathfrak{M} is atomic, there is a principal formula $\phi(x, y)$ isolating $tp(\bar{c}, \bar{b})$. Therefore, from indiscernibility, $\phi(\bar{x}, \bar{a})$ is a principal formula such that $\mathfrak{N} \models \forall \bar{x} (\phi(\bar{x}, \bar{a}) \rightarrow \theta(\bar{x}, \bar{a}))$. \square

Lemma 2. *Let \mathfrak{N} be a model of T and $X \subseteq V^{\mathfrak{N}}$. If \mathfrak{N} is atomic over X , then*

- (i) $|U^{\mathfrak{N}}| \geq |A|$;
- (ii) $\text{cl}(Y)$ is finite for all finite $Y \subseteq U^{\mathfrak{N}}$;
- (iii) there is no cl-independent subset of size 3 in $U^{\mathfrak{N}}$.

Proof. (i) $|U^{\mathfrak{N}}| \geq |X|$ because for each $x \in X$ $p^{-1}x$ is non-empty.

(ii) As \mathfrak{N} is atomic, $tp(Y|X)$ is isolated by some formula $\theta(\bar{y}, \bar{c})$, where \bar{c} is a k tuple of distinct elements from X . As V is indiscernable, $\theta(\bar{y}, \bar{b})$ is principal for any k tuple \bar{b} of distinct elements from \mathfrak{M} . Choose \bar{d} from \mathfrak{M} realizing $\theta(\bar{y}, \bar{b})$ and suppose that $|\text{cl}(\bar{d})| = l < \omega$. Then as $\theta(\bar{y}, \bar{d})$ is principal, $\theta(\bar{y}, \bar{d})$ implies $|\text{cl}(\bar{y})| \leq l$.

(iii) Assume that $c_0, c_1, c_2 \in U^{\mathfrak{N}}$ are cl-independent. As \mathfrak{N} is atomic over X $tp(\bar{c})$ is principal, so let $\theta(\bar{x}, \bar{a})$ isolate $tp(\bar{c})$. Choose \bar{b} from $V^{\mathfrak{M}}$ and \bar{d} from $U^{\mathfrak{M}}$ such that $\mathfrak{M} \models \theta(\bar{d}, \bar{b})$. But, then \bar{d} is a cl-independent subset of $U^{\mathfrak{M}}$, which is a contradiction. □

An abstract closure relation on a set X is a function $\text{cl} : \wp(X) \rightarrow \wp(X)$ such that for all subsets A and B of X and all $b \in X$, $A \subseteq \text{cl}(A)$, $\text{cl}(\text{cl}(A)) = \text{cl}(A)$, $A \subseteq B$ implies $\text{cl}(A) \subseteq \text{cl}(B)$, and $b \in \text{cl}(A)$ implies there is a finite subset A_0 of A such that $b \in \text{cl}(A_0)$. The following combinatorial lemma is known [4]. It can be proved by induction on n .

Lemma 3. *For all ordinals α and all $n \in \omega$, if $|X| \geq \aleph_{\alpha+n}$ and cl is a closure relation on X such that $|\text{cl}(A)| < \aleph_{\alpha}$ for all finite subsets A of X , then X contains a cl-independent subset of size $n + 1$.*

Theorem 4. *Let X be a subset of $V^{\mathfrak{N}}$ for an arbitrary model \mathfrak{N} of T . Then, the principal types over X are dense, but there is an atomic model over X iff $|X| \leq \aleph_1$.*

If $|X| \leq \aleph_1$, then it is known [9] that there is an atomic model over X . However, if $|X| \geq \aleph_2$, then there cannot be an atomic model over X by Lemmas 2 and 3.

We now translate the result to cylindric algebras. We write $Is(\mathfrak{A}, \mathfrak{B})$ for the set of all isomorphisms from \mathfrak{A} to \mathfrak{B} . Let T be a first order complete first order theory and let \mathfrak{Fm}_T be the quotient algebra. Then for any $\mathfrak{A} \in \mathbf{Cs}_\omega^{reg} \cap \mathbf{Lf}_\omega$ and $h \in Is(\mathfrak{Fm}_T, \mathfrak{A})$, there is a model \mathfrak{M} for \mathcal{L} such that $\mathfrak{A} = \mathbf{Cs}_n^{\mathfrak{M}}$ and $h(\phi / \equiv_T) = \phi^{\mathfrak{M}}$. Conversely, for any \mathfrak{M} of \mathcal{L} and $\mathfrak{A} = \mathbf{Cs}_n^{\mathfrak{M}}$, and there is an $h \in Is(\mathfrak{Fm}_T, \mathfrak{A})$ such that $h(\phi / \equiv_T) = \phi^{\mathfrak{M}}$. We refer to [7] [4.3] for such equivalences.

Lemma 1 translate to $\mathfrak{Nr}_n(\mathbf{Cs}_n^{\mathfrak{M}})_X$ is atomic for all n . Theorem 4 says that \mathfrak{N} is atomic over X iff $|X| \leq \aleph_1$, that is,

$$1 = \bigcup \{x : x \in At\mathfrak{Nr}_n(\mathbf{Cs}_n^{\mathfrak{M}})_X\} \text{ iff } |X| \leq \aleph_1.$$

The main theorem follows.

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